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NUMBER THEORETICAL PECULIARITIES IN THE DIMENSION THEORY OF DYNAMICAL SYSTEMS

BY

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ABSTRACT

We show that dimensional theoretical properties of dynamical systems can considerably change because of number theoretical peculiarities of some parameter values.

1. Introduction

In the last decades there has been enormous interest in geometrical invariants of dynamical systems especially in the Hausdorff dimension of invariant sets like attractors, repellers or hyperbolic sets and ergodic measures on these sets. A dimension theory of dynamical systems was developed and nowadays the Hausdorff dimension seems to have its place beside classical invariants like entropy or Lyapunov exponents.**

There are two main principles that form a kind of a guideline through the dimension theory of dynamical systems. The first states the identity of the Hausdorff and box-counting dimension of invariant sets. The second one is the variational principle for Hausdorff dimension which states that the Hausdorff dimension of a given invariant set can be approximated by the Hausdorff dimension

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** We refer to the book of Falconer [6] for an introduction to dimension theory and recommend the book of Pesin [17] for the dimension theory of dynamical systems.

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of ergodic measures on these sets or in a stronger form states the existence of an ergodic measure of full Hausdorff dimension on an given invariant set. In many situations these principles are essential to determine the Hausdorff dimension of an invariant set and for relating this quantity to other characteristics of the dynamics like entropy, Lyapunov exponents and pressure.

For conformal repellers we know that the identity of Hausdorff and box-counting dimension holds and that there exists an ergodic measure of full Hausdorff dimension (see chapter 7 of [17]). For hyperbolic sets of diffeomorphisms the variational principle for Hausdorff dimension does not hold in general (see [13]). But again, if the system is conformal restricted to stable resp. unstable manifolds, there exists an ergodic measure of full dimension for the restrictions and the identity of box-counting and Hausdorff dimension of the hyperbolic set holds (see again chapter 7 of [17]). In the non-conformal situation there is no general theory this days that allows us to determine the dimensional theoretical properties of a given dynamical system. But there are a lot of results for special classes of systems that state that the variational principle or the identity of box-counting and Hausdorff dimension or both hold at least generically in the sense of Lebesgue measure on the parameter space (see for instance [5], [21], [15], [24], [23]). In this paper we focus on such classes of systems.

We will show that in situations where there generically exists an ergodic measure of full Hausdorff dimension, the variational principle for Hausdorff dimension may not hold in general because of a number theoretical peculiarities of some parameter values (see Theorem 2.1 below). Furthermore, we will show that the identity of box-counting and Hausdorff dimension may drop because of number theoretical peculiarities in situations where this identity generically holds (see Theorem 2.2 below). Our example for the first phenomena is the Fat Baker's transformation and our example for the second phenomena is a class of self-affine repellers. Both classes of systems are very simple, but it seems obvious to us that the same phenomena appear as well in more complicated examples; also, this would be of course harder to prove.

All our results are related to a special class of algebraic integers, namely Pisot–Vijayarghavan numbers* (in brief, PV numbers), and they are in some sense the consequence of a generalization of results of Erdös [4], Garsia ([7], [8]) and Alexander and Yorke [1] on the singularity and dimension of conveniently convolved measures which has been proved by Lalley [12]. We think that from the viewpoint of geometric measure theory and algebraic number theory this result

* See Appendix B at the end of this work.

is interesting in itself (see Theorem 4.1 below).

The rest of the paper is organized as follows. In section 2 we define the systems we study, state our main theorems about these systems and comment on our results. In section 3 we introduce coding maps for our systems and find representations of all ergodic measures using these codings. In section 4 we define a class of Borel probability measures associated with PV numbers (Erdös measures), introduce a kind of entropy related to this measure (Garsia entropy) and state the main theorem about the singularity, the Garsia entropy and the Hausdorff dimension of Erdös measures. The proof of Theorem 2.1 is contained in section 5 and the proof of Theorem 2.2 can be found in section 6.

In Appendix A we collect some basic facts in dimension theory and in Appendix B we define PV numbers and present a table with examples of these algebraic integers.

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2. Basic definitions and main results

For $\beta \in (0.5, 1)$ we define the **Fat Baker's transformation** $f_\beta\colon \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$ by

$$f_\beta(x,y) = \begin{cases} (\beta x + (1-\beta), 2y-1) & \text{if } y \geq 0, \\ (\beta x - (1-\beta), 2y+1) & \text{if } y < 0. \end{cases}$$

This map was introduced by Alexander and Yorke in [1]. It is called Fat Baker's transformation because if we set $\beta = 0.5$ we get the classical Baker's transformation.

It is obvious that the attractor of f_β is the whole square $[-1,1]^2$ which has Hausdorff and box-counting dimension two. We always restrict f_β to its attractor.

Now we state our main result about the Fat Baker's transformation.

THEOREM 2.1: *If $\beta \in (0.5,1)$ is the reciprocal of a PV number, then the variational principle for Hausdorff dimension does not hold for $([-1,1]^2, f_\beta)$, i.e., $\{\dim_H \mu\colon \mu\ f_\beta\text{-ergodic}\} < 2$.*

Remark 2.1: Theorem 2.1 is an extension of the result of Alexander and Yorke [1] which states that the Sinai–Ruelle–Bowen measures for $([-1,1]^2, f_\beta)$ do not have full Rényi dimension.

Remark 2.2: It follows from [1] together with Solomyak's theorem about Bernoulli convolutions [22] that for almost all $\beta \in (0.5, 1)$ the Sinai-Ruelle-Bowen measures for $([-1,1]^2, f_\beta)$ have full dimension. Thus our theorem shows that in situations where generically there is an ergodic measure of full dimension, the variational principle for Hausdorff dimension may not hold in general because of special number theoretical properties of some parameter values. As far as we know our theorem provides the first example of this type.

Now we come to our second class of examples. For $\beta \in (0.5, 1)$ and $\tau \in (0, 0.5)$ we define two affine contractions on $[-1,1]^2$ by

$$T_1^{\beta,\tau}(x,z) = (\beta x + (1-\beta), \tau z + (1-\tau)),$$
$$T_{-1}^{\beta,\tau}(x,z) = (\beta x - (1-\beta), \tau z - (1-\tau)).$$

From [10] we know that there is a unique compact self-affine subset $\Lambda_{\beta,\tau}$ of $[-1,1]^2$ satisfying

$$\Lambda_{\beta,\tau} = T_1^{\beta,\tau}(\Lambda_{\beta,\tau}) \cup T_{-1}^{\beta,\tau}(\Lambda_{\beta,\tau}).$$

Let $T_{\beta,\tau}$ be the smooth expanding transformation on

$$T_1^{\beta,\tau}([-1,1]^2) \cup T_{-1}^{\beta,\tau}([-1,1]^2)$$

defined by

$$T_{\beta,\tau}(x) = (T_i^{\beta,\tau})^{-1}(x) \quad \text{if } x \in T_i^{\beta,\tau}([-1,1]^2) \quad \text{for } i = 1, -1.$$

Obviously the set $\Lambda_{\beta,\tau}$ is an invariant repeller for the transformation $T_{\beta,\tau}$. We call the system $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ a **self-affine repeller**.

Let us state our main result about the systems $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$.

THEOREM 2.2: *Let $\beta \in (0.5, 1)$ be the reciprocal of a PV number. For all $\tau \in (0, 0.5)$ we have $\dim_H \Lambda_{\beta,\tau} < \dim_B \Lambda_{\beta,\tau}$. Moreover, if τ is sufficiently small there cannot be a Bernoulli measure of full dimension for the system $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$.*

Remark 2.3: We know from [15] that for almost all $\beta \in (0,5,1)$ and all $\tau \in (0, 0.5)$ the identity

$$\dim_H \Lambda_{\beta,\tau} = \dim_B \Lambda_{\beta,\tau} = \frac{\log 2\beta}{\log \tau} + 1$$

holds and that there is a Bernoulli measure of full dimension for $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$. Thus Theorem 2.2 shows that dimensional theoretical properties of dynamical systems can considerably change because of number theoretical peculiarities.

Remark 2.4: The fact that the identity of Hausdorff and box-counting dimension may drop because of number theoretical peculiarities was shown before by Przytycki and Urbanski [19] in the context of Weierstrass like functions. Pollicott and Wise [18] claimed (without proof) that the first statement of our theorem follows for small τ from the work of Przytycki and Urbanski. We were not able to see that this is true and thus wrote down an independent proof which gives explicit upper bounds on $\dim_H \Lambda_{\beta,\tau}$ (see section 6).

Remark 2.5: We would like to mention that another interesting class of self-affine sets (so called McMullen carpets) was analyzed by McMullen [14] and in a more general setting by Gatzouras and Lalley [9]. Their result shows that the identity of box-counting and Hausdorff dimension is not generic but exceptional for these sets. On the other hand, the existence of a Bernoulli measure of full Hausdorff dimension holds generally for McMullen carpets and allows the calculation of the Hausdorff dimension of these sets. Also, the construction of McMullen carpets is similar to the construction of the sets considered in our work. The dimensional theoretical properties of the two classes of systems are very different and number theoretical peculiarities do not play a role in the context of McMullen carpets.

Remark 2.6: We do not know if there exists an ergodic measure of full Hausdorff dimension for the systems $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ and we cannot calculate $\dim_H \Lambda_{\beta,\tau}$ in the case that $\beta \in (0.5, 1)$ is the reciprocal of a PV number. The second statement of our theorem only shows that it is not possible to calculate $\dim_H \Lambda_{\beta,\tau}$ by means of Bernoulli measures in this situation.

3. Coding maps and representation of ergodic measures

We first introduce here the symbolic spaces which we use for our coding. Let $\Sigma = \{-1,1\}^{\mathbb{Z}}$ and $\Sigma^+ = \{-1,1\}^{\mathbb{N}_0}$. By pr_+ we denote the projection from Σ onto Σ^+. With a natural product metric Σ (resp. Σ^+) comes a perfect, totally disconnected and compact metric space. For $u, v \in \mathbb{Z}$ (resp. $u, v \in \mathbb{N}$) and $t_0, t_1, \ldots, t_u \in \{-1, 1\}$ we define a cylinder set in Σ (resp. Σ^+) by

$$[t_0, t_1, \ldots, t_u]_v := \{(s_k) \colon s_{v+k} = t_k \text{ for } k = 0, \ldots, u\}.$$

The cylinder sets form a basis for the metric topology on Σ (resp. Σ^+). The forward shift map σ on Σ (resp. Σ^+) is given by $\sigma((s_k)) = (s_{k+1})$. The backward shift σ^{-1} is defined on Σ and given by $\sigma((s_k)) = (s_{k-1})$. By b^p for $p \in (0,1)$ we denote the Bernoulli measure on Σ (resp. Σ^+), which is the product of the discrete measure giving 1 the probability p and -1 the probability $(1-p)$. We write b for the equal-weighted Bernoulli measure $b^{0.5}$. The Bernoulli measures are ergodic with respect to forward and backward shifts (see [3]).

We are now prepared to define the Shift coding for the Fat Baker's transformation $([-1,1]^2, f_\beta)$. Define a continuous map $\hat{\pi}_\beta$ from Σ onto $[-1,1]^2$ by

$$\hat{\pi}_\beta(\underline{i}) = ((1-\beta)\sum_{k=0}^{\infty} i_k\beta^k, \sum_{k=1}^{\infty} i_{-k}(1/2)^k).$$

A simple check shows that

$$f_\beta \circ \hat{\pi}_\beta(\underline{i}) = \hat{\pi}_\beta \circ \sigma^{-1}(\underline{i}) \quad \forall \underline{i} \in \bar{\Sigma} = (\Sigma \backslash \{(s_k)\colon \exists k_0 \forall k \leq k_0 : s_k = 1\}) \cup \{(1)\}.$$

Note that if μ is a σ-invariant Borel probability measure on Σ we have $\mu(\bar{\Sigma}) = 1$. From this fact, by applying standard techniques in ergodic theory it is possible to show that the map

$$\mu \longmapsto \mu_\beta := \mu \circ \hat{\pi}_\beta^{-1}$$

from the space of σ-ergodic Borel probability measures on Σ is continuous with respect to the weak* topology and is onto the space of f_β-ergodic Borel probability measures on $[-1,1]^2$. Moreover, the system $([-1,1], f_\beta, \mu_\beta)$ is a measure theoretical factor of $(\Sigma, \sigma^{-1}, \mu)$.

Now we introduce a shift coding for the self-affine repeller $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$. Consider the homeomorphism $\pi_{\beta,\tau}\colon \Sigma^+ \to \Lambda_{\beta,\tau}$ given by

$$\pi_{\beta,\tau}(\underline{i}) = \left((1-\beta)\sum_{k=0}^{\infty} i_k\beta^k, (1-\tau)\sum_{k=0}^{\infty} i_k\tau_k\right).$$

It is easy to see that $\pi_{\beta,\tau} \circ \sigma = T_{\beta,\tau} \circ \sigma$. Thus the system $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ is homeomorph conjugated to (Σ, σ) and the map

$$\mu \longmapsto \mu_{\beta,\tau} := \mu \circ \pi_{\beta,\tau}^{-1}$$

is a homeomorphism with respect to the weak* from the space of σ-ergodic Borel probability measures on Σ^+ onto the space of $T_{\beta,\tau}$-ergodic Borel probability measures on $\Lambda_{\beta,tau}$.

4. Erdös measures and Garsia entropy

For $\beta \in (0.5, 1)$ define a continuous map from Σ^+ onto $[-1,1]$ by

$$\pi_\beta(\underline{i}) = (1-\beta)\sum_{k=0}^{\infty} i_k\beta^k.$$

Given a Borel probability measure ν on Σ^+ we define a Borel probability measure on $[-1,1]$ by $\nu_\beta = \nu \circ \pi_\beta^{-1}$. If we choose the Bernoulli measure b^p on Σ^+ for a $p \in (0,1)$, then b_β^p is a self-similar measure which is usually called a Bernoulli convolution. There are many results in the literature about Bernoulli convolutions and we cannot cite all of them here. Instead we refer to the nice overview article by Peres, Solomyak and Schlag [16].

In our work we are not only interested in Bernoulli convolutions but in all measures ν_β where ν is a σ invariant Borel probability measure Σ^+ and $\beta \in (0.5,1)$ is the reciprocal of a PV number (see Appendix B). We call a measure of this type an **Erdös** measure.

Now we introduce a special kind of entropy related to Erdös measure. What we do here is generalize the approach of Garsia ([7], [8]) for Bernoulli convolutions to all Erdös measures. Let $\sim_{n,\beta}$ be the equivalence relation on Σ^+ given by

$$\underline{i} \sim_{n,\beta} \underline{j} \Leftrightarrow \sum_{k=0}^{n-1} i_k \beta^k = \sum_{k=0}^{n-1} j_k \beta^k$$

and define a partition $\Pi_{n,\beta}$ of Σ^+ by $\Pi_{n,\beta} = \Sigma^+/\sim_{n,\beta}$. Recall that entropy of a partition Π with respect to a Borel probability measure ν on Σ^+ is

$$H_\nu(\Pi) = -\sum_{P\in\Pi} \nu(P)\log\nu(P).$$

We denote the join of two partitions Π_1 and Π_2 by $\Pi_1 \vee \Pi_2$. This is the partition consisting of all sections $A \cap B$ for $A \in \Pi_1$ and $B \in \Pi_2$. It is easy to see that $\Pi_{n,\beta} \vee \sigma^{-n}(\Pi_{m,\beta})$ is finer than the partition $\Pi_{n+m,\beta}$ and hence the sequence $H_\nu(\Pi_{n,\beta})$ is sub-additive for a shift invariant measure ν on Σ^+. We can thus define the **Garsia entropy** $G_\beta(\nu)$ for a shift invariant Borel probability measure ν on Σ^+ by

$$G_\beta(\nu) := \lim_{n\longrightarrow\infty} \frac{H_\nu(\Pi_{n,\beta})}{n} = \inf_n \frac{H_\nu(\Pi_{n,\beta})}{n}.$$

The limit exists and is equal to the infimum since the sequence $H_\nu(\Pi_{n,\beta})$ is sub-additive. Another simple consequence of the sub-additivity of this sequence is that the map

$$\nu \longmapsto G_\beta(\nu)$$

is upper-semi-continuous with respect to the weak* topology on the space of σ invariant Borel probability measures on Σ^+.

We are now prepared to state the main theorem about Erdös measures and Garsia entropy which is essentially based on the work of Lalley [12].

THEOREM 4.1: *Let $\beta \in (0.5, 1)$ be the reciprocal of a PV number. For all σ-ergodic Borel probability measures ν on Σ^+ the following equivalence holds:*

$$\nu_\beta \text{ is singular} \Leftrightarrow G_\beta(\nu) < -\log\beta \Leftrightarrow \dim_H \nu_\beta < 1.$$

Moreover, the set of σ-ergodic measures Borel probability measures ν on Σ^+ such that ν_β is singular is open in the weak topology and contains the Bernoulli measures b^p for $p \in (0, 1)$.*

Proof: From Proposition 3 of [12] we know that

$$\dim_H \nu_\beta < -\frac{G_\beta(\nu)}{\log\beta}$$

and from Proposition 5 of [12] we have

$$\nu_\beta \text{ singular} \Rightarrow \dim_H \nu_\beta < 1.$$

Now note that $\dim_H \nu_\beta < 1$ obviously implies the singularity of ν_β. Thus we get the equivalence stated in Theorem 4.1.

Now choose a singular Erdös measure ξ_β. We have $G_\beta(\xi) < \log\beta^{-1}$. By upper-semi-continuity of G we get $G_\beta(\nu) < \log\beta^{-1}$ and hence $\dim \nu_\beta < 1$ for all ν in a hole weak* neighborhood of ξ. Thus the set $\{\nu\colon \nu_\beta \text{ is singular}\}$ is open in the weak* topology.

It has been shown by Erdös [4] that the equal-weighted Bernoulli convolution b_β is singular if $\beta \in (0.5, 1)$ is the reciprocal of a PV number and the argument has been extended to all Bernoulli convolutions in Proposition 2 of [12]. ∎

Remark 4.1: Using the result of Erdös [4] about the singularity of the equal-weighted Bernoulli convolution b_β, Garsia [7] proved $G_\beta(b) < -\log\beta$. From this, Alexander and Yorke [1] deduced that the Rénji dimension of b_β is less than one.

Remark 4.2: The PV case is exceptional. It was shown by Solomyak [22] that for almost all $\beta \in (0.5, 1)$ the Bernoulli convolution b_β is absolutely continuous with density in L^2.

5. Proof of Theorem 2.1

The proof of Theorem 2.1 follows from Theorem 4.1 and two propositions providing upper estimates on the Hausdorff dimension of all ergodic measures μ_β for the Fat Baker's transformation f_β. It can be found at the end of this section.

PROPOSITION 5.1: *If μ is a shift ergodic Borel probability measure on Σ and $\beta \in (0.5, 1)$, we have*

$$\dim_H \mu_\beta \leq 1+ \leq G_\beta(pr_+(\mu))/-\log\beta$$

where pr_+ denotes the projection from Σ onto Σ^+.

Proof: By Proposition A1 of Appendix A and the definition of the Hausdorff dimension of a measure, we have $\dim_H \mu_\beta \leq 1+\dim_H pr_X \mu_\beta$ where pr_X denotes the projection onto the first coordinate axis. Just by the definition of the involved measures we have $pr_X \mu_\beta = (pr_+\mu)_\beta$ and hence $\dim_H \mu_\beta \leq 1+\dim_H (pr_+\mu)_\beta$. The proposition follows now immediately from Proposition 3 of [12]. ∎

PROPOSITION 5.2: *If μ is a shift ergodic Borel probability measure on Σ and $\beta \in (0.5, 1)$, we have*

$$\dim_H \mu_\beta \leq 1+ \leq h_\mu(\sigma)/\log 2$$

where $h_\mu(\sigma)$ is the usual measure-theoretic entropy of the shift (Σ, σ, μ).

Proof: By Proposition A1 and the definition of the Hausdorff dimension of a measure, we have $\dim_H \mu_\beta \leq 1+\dim_H pr_Y \mu_\beta$ where pr_Y denotes the projection onto the second coordinate axis. By definition, we have $pr_Y \mu_\beta = \mu_\beta \circ pr_Y^{-1} = \mu \circ \hat{\pi}_\beta^{-1} \circ pr_Y^{-1}$. By the properties of the coding map $\hat{\pi}_\beta$ it is easy to check that this measure is ergodic with respect to the map $f\colon [-1,1] \longmapsto [-1,1]$ given by

$$f(y) = \begin{cases} 2y-1 & \text{if } y \geq 0, \\ 2y+1 & \text{if } y < 0. \end{cases}$$

Thus the Hausdorff dimension of $pr_Y \mu_\beta$ is well known (see [17]),

$$\dim_H pr_Y \mu_\beta = \frac{h_{pr_Y \mu_\beta}(f)}{\log 2}.$$

Moreover, we know that $([-1,1], f, pr_Y \mu_\beta)$ is a measure theoretical factor of $([-1,1]^2, f_\beta, \mu_\beta)$ and that this system is a factor of (Σ, σ, μ). Hence we get by well known properties of the entropy (see [3]) $h_{pr_Y \mu_\beta}(f) \leq h_\mu(\sigma)$, which completes the proof. ∎

Proof of Theorem 2.1: From Theorem 4.1 and the upper-semi-continuity of G_β we get $G_\beta(pr^+\mu)/\log\beta^{-1} \leq c_1 < 1$ for all μ in hole weak* neighborhood U of b in the space of σ-ergodic Borel probability measures on Σ. Hence by Proposition

6.1, $\dim_H \bar{\mu}_\beta \leq c_1+1 < 2$ holds for all μ in U. On the other hand, we have by well-known properties of the measure theoretical entropy, $h_\mu(\sigma)/\log 2 \leq c_2 < 1$ on the complement of U (see [3]). From Proposition 6.1 we thus get $\dim_H \mu_\beta \leq c_2+1 < 2$ for all μ in the complement of U. Putting these facts together we obtain

$$\dim_H \mu_\beta \leq \max\{c_1, c_2\} + 1 < 2 = \dim_H[-1,1]^2.$$

But we know that all ergodic measures for the system $([-1,1]^2, f_\beta)$ are of the form μ_β for some σ-ergodic Borel probability measures μ on Σ, and the proof is complete. ∎

6. Proof of Theorem 2.2

The proof of Theorem 2.2 has a lot of ingredients, a formula for $\dim_B \Lambda_{\beta,\tau}$ found in [18], a formula for $\dim_H b^p_{\beta,\tau}$ found in [15], Theorem 4.1 and the following two propositions giving upper bounds on $\dim_H \Lambda_{\beta,\tau}$.

PROPOSITION 6.1: *If $\beta \in (0.5, 1)$ is the reciprocal of an PV number and $\tau \in (0, 0.5)$, we have*

$$\dim_H \Lambda_{\beta,\tau} \leq \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\log\beta/\log\tau})}{n \log \beta^{-1}} \qquad \forall n \geq 1$$

where $\Pi_{n,\beta}$ is the partition of Σ^+ defined in section 4 and $\sharp P$ denotes the number of cylinder sets of length n contained in an element of this partition.

Proof: Fix a reciprocal of a PV number $\beta \in (0.5, 1)$ and $\tau \in (0, 0.5)$. Let $n \geq 1$ and set

$$u_n = \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\log\beta/\log\tau})}{n \log \beta^{-1}}.$$

Consider the set of cylinders in Σ^+ given by

$$C_n = \{[\tilde{s}_1\tilde{s}_2 \ldots \tilde{s}_m]_0 \colon \tilde{s}_i \in \{-1,1\}^n, i = 1, \ldots, m\}.$$

Define a set function η on C_n by

$$\eta([\tilde{s}]_0) = \frac{\sharp P(\tilde{s})^{\log\beta/\log\tau}}{\sharp P(\tilde{s})} \beta^{n u_n} \quad \text{and}$$
$$\eta([\tilde{s}_1\tilde{s}_2 \ldots \tilde{s}_m]_0) = \eta([\tilde{s}_1]_0) \cdot \eta([\tilde{s}_2]_0) \cdot \ldots \cdot \eta([\tilde{s}_m]_0)$$

where $\tilde{s}, \tilde{s}_1, \ldots, \tilde{s}_m$ are elements of $\{-1,1\}^n$ and $P(\tilde{s})$ denotes the element of the partition $\Pi_{n,\beta}$ containing the cylinder $[\tilde{s}]_0$.

Note that C_n is a basis of the metric topology of Σ^+ and that $\sum_{\tilde{s}\in\{-1,1\}^n}\eta([\tilde{s}]_0) = 1$ by the definition of u_n. Thus we can extend η to a Borel probability measure on Σ^+ and $\eta_{\beta,\tau} := \eta\circ\pi_{\beta,\tau}^{-1}$ defines a Borel probability measure on $\Lambda_{\beta,\tau}$.

Given $m \geq 1$ we set $q(m) = \lceil m(\log\beta/\log\tau)\rceil$. Given a $\tilde{s}_i \in \{-1,1\}^n$ for $i = 1,\ldots,m$ we define a subset of $\Lambda_{\beta,\tau}$ by

$$\begin{aligned} R_{\tilde{s}_1\ldots\tilde{s}_n} = \{(\sum_{i=0}^{\infty} s_i(1-\beta)\beta^i, \sum_{i=0}^{\infty} t_i(1-\tau)\tau^i)\colon s_i,t_i\in\{-1,1\} \\ (s_{(i-1)n},\ldots,s_{in-1}) = \tilde{s}_i \quad i=1,\ldots,m \quad \text{and} \\ (t_{(i-1)n},\ldots,t_{in-1}) = \tilde{s}_i \quad i=1,\ldots,q(m)\}. \end{aligned}$$

We see that $R_{\tilde{s}_1\ldots\tilde{s}_m}$ is "almost" a square in $\Lambda_{\beta,\tau}$ of side length β^{mn}. More precisely we have

$$c_1\beta^{mn} \leq \mathrm{diam} R_{\tilde{s}_1\ldots\tilde{s}_m} \leq c_2\beta^{mn} \tag{1}$$

where the constants c_1, c_2 are independent of the choice of $\tilde{s}_i$.

Now let us examine the $\eta_{\beta,\tau}$ measure of the sets $R_{\tilde{s}_1,\ldots,\tilde{s}_m}$.

Assume that $\tilde{t}_i \sim_{n,\beta} \tilde{s}_i$ for $i = q(m)+1,\ldots,m$ where $\sim_{n,\beta}$ is the equivalence relation introduced in section 4. The rectangles $\pi_{\beta,\tau}([\tilde{s}_1\ldots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\ldots\tilde{t}_m]_0)$ are all disjoint and lie above each other in the set $R_{\tilde{s}_1\ldots\tilde{s}_m}$. Hence we have

$$\begin{aligned} \eta_{\beta,\tau}(R_{\tilde{s}_1\ldots\tilde{s}_m}) &\geq \eta\Bigg(\bigcup_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \ i=q(m)+1,\ldots,m} \pi_{\beta,\tau}([\tilde{s}_1\ldots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\ldots\tilde{t}_m]_0)\Bigg) \\ &= \sum_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \ i=q(m)+1,\ldots,m} \eta([\tilde{s}_1\ldots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\ldots\tilde{t}_m]_0). \end{aligned}$$

Using the fact $\tilde{s}\sim_{n,\beta}\tilde{t} \Rightarrow \sharp P(\tilde{s}) = \sharp P(\tilde{t}) \Rightarrow \eta([\tilde{s}]_0) = \eta([\tilde{t}]_0)$, this last expression equals

$$\begin{aligned} &\prod_{i=1}^{m}\eta([\tilde{s}_i]_0) \sum_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \ i=q(m)+1,\ldots,m} 1 \\ &= \prod_{i=1}^{m}\frac{\sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\sharp P(\tilde{s}_i)}\beta^{mnu_n} \sum_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \ i=q(m)+1,\ldots,m} 1 \\ &= \frac{\prod_{i=1}^{m}\sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\prod_{i=1}^{q(m)}\sharp P(\tilde{s}_i)}\beta^{mnu_n} = (\phi_{\tilde{s}_1\ldots\tilde{s}_m}\beta^{nu_n})^m \end{aligned}$$

where

$$\phi_{\tilde{s}_1\ldots\tilde{s}_m} = \left(\frac{\prod_{i=1}^{m}\sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\prod_{i=1}^{q(m)}\sharp P(\tilde{s}_i)}\right)^{1/m}.$$

Now fix an $\epsilon > 0$. We use the sets $R_{\tilde{s}_1 \dots \tilde{s}_m}$ to construct a good cover of $\Lambda_{\beta,\tau}$ in the sense for Hausdorff dimension. To this end set

$$R_m := \{R_{\tilde{s}_1 \dots \tilde{s}_m} \colon \phi_{\tilde{s}_1 \dots \tilde{s}_m} \geq \beta^{n\epsilon}\}.$$

We have an upper bound on the cardinality of R_m. If $R \in R_m$ then $\eta_{\beta,\tau}(R) \geq \beta^{mn(u_n+\epsilon)}$ and since $\eta_{\beta,\tau}$ is a probability measure we see that

$$\operatorname{card}(R_m) \leq \beta^{-mn(u_n+\epsilon)}. \tag{2}$$

Now let $R(M) = \bigcup_{m \geq M} R_m$. We want to prove that $R(M)$ is a cover of $\Lambda_{\beta,\tau}$ for all $M \geq 1$.

For $\underline{s} = (s_k) \in \Sigma^+$ we define the function ϕ_m by $\phi_m(\underline{s}) = \phi_{s_0 \dots s_{mn-1}}$. In addition, we need two auxiliary functions on Σ^+:

$$f_m(\underline{s}) = \frac{\prod_{i=0}^{m} \sharp P((s_{(i-1)n}, \dots, s_{in-1}))^{1/m}}{\prod_{i=0}^{q(m)} \sharp P((s_{(i-1)n}, \dots, s_{in-1}))^{1/q(m)}},$$

$$g_m(\underline{s}) = \left(\prod_{i=1}^{q(m)} \sharp P((s_{(i-1)n}, \dots, s_{in-1})) \right)^{1/q(m)(\log\beta \log\tau - q(m)/m)}.$$

Since $1 \leq \sharp P(\tilde{s}) \leq 2^n$ we have $1 \leq g_m(\underline{s}) \leq 2^{n(\log\beta/\log\tau - q(m)/m)}$. Thus by the definition of $q(m)$ we have $g_m(\underline{s}) \longrightarrow 1$. Moreover, we have $\overline{\lim}_{m \longrightarrow \infty} f_m(\underline{s}) \geq 1$ because $\prod_{i=0}^{t} \sharp P((s_{i-1}n, \dots, s_{in-1}))^{1/t} \geq 1 \; \forall t \geq 1$.

A simple calculation shows $\phi_m(\underline{s}) = (f_m(\underline{s}))^{\log\beta/\log\tau} g_m(\underline{s})$. The properties of f and g thus imply

$$\overline{\lim}_{m \longrightarrow \infty} \phi_m(\underline{s}) \geq 1 \quad \forall \, \underline{s} \in \Sigma^+.$$

This will help us to show that $R(M)$ is a cover of $\Lambda_{\beta,\tau}$. For all $\underline{s} = (s_k) \in \Sigma^+$ there is an $m \geq M$ such that $\phi_m(\underline{s}) \geq \beta^{n\epsilon}$ and thus $\pi_{\beta,\tau}(\underline{s}) \in R_{s_0,\dots,s_{mn-1}} \in R(M)$. Since $\pi_{\beta,\tau}$ is onto $\Lambda_{\beta,\tau}$ we see that $R(M)$ is indeed a cover of $\Lambda_{\beta,\tau}$.

We are now able to complete the proof. For every $\epsilon > 0$ and every $M \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{R \in R(M)} (\operatorname{diam} R)^{u_n+2\epsilon} &= \sum_{m \geq M} \sum_{R \in R_m} (\operatorname{diam} R)^{u_n+2\epsilon} \\ \leq^{(1)} \sum_{m \geq M} \sum_{R \in R_m} (c_2 \beta^{mn})^{u_n+2\epsilon} &= \sum_{m \geq M} \operatorname{card}(R_m)(c_2\beta^{mn})^{u_n+2\epsilon} \\ &\leq^{(2)} c_2^{u_n+2\epsilon} \sum_{m \geq M} \beta^{mn\epsilon}. \end{aligned}$$

The last expression goes to zero as $M \longrightarrow 0$. By the definition for Hausdorff dimension we thus get $\dim_H \Lambda_{\beta,\tau} \leq u_n + 2\epsilon$ and, since ϵ is arbitrary, we have $\dim_H \Lambda_{\beta,\tau} \leq u_n$. ∎

Remark 6.1: Some ideas we used here are due to the proof of McMullen's theorem on self-affine carpets [14] by Pesin in [17].

Now we use properties of PV numbers to get the following estimate:

PROPOSITION 6.2: *If $\beta \in (0.5, 1)$ is the reciprocal of a PV number and $\tau \in (0, 0.5)$, we have*

$$\exists N \in \mathbb{N}\ \forall n > N \quad \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\log\beta/\log\tau})}{n\log\beta^{-1}} < \frac{\log(2\beta/\tau)}{\log(1/\tau)}.$$

Proof: Fix β. Define π_n from Σ^+ to $[-1,1]$ by $\pi_n((s_k)) = \sum_{k=0}^{n-1} s_k(1-\beta)\beta^k$ and let $b_n = b \circ \pi_n^{-1}$. Let $\sharp(n)$ be the number of distinct points of the form $\sum_{k=0}^{n-1} \pm(1-\beta)\beta^k$ and $\omega(n)$ be the minimal distance between two of these points. Furthermore, denote the points by x_i^n, $i = 1, \ldots, \sharp(n)$ and let P_n^i be the corresponding element in $\Sigma_{n,\beta}$.

Since b_β is singular, we have $\forall C \in (0,1)\ \forall \epsilon > 0\ \exists$ disjoint intervals $(a_1, b_1), \ldots, (a_u, b_u)$ with

$$\sum_{l=1}^{u} (b_l - a_l) < \epsilon \quad \text{and} \quad b_\beta(O) > C \text{ where } O := \bigcup_{l=0}^{u} (a_l, b_l).$$

Without loss of generality we may assume $b_\beta(a_l) = b_\beta(b_l) = 0$ for $l = 1, \ldots, u$. It is obvious that the discrete distribution b_n converges weakly to b_β. Thus we have $\exists n_1(\epsilon, C)\ \forall n > n_1(\epsilon, C)$: $b_n(O) > C$. We now expand the intervals a little bit, so that their length is a multiple of $\omega(n)$.

$$\begin{aligned} k_{l,n} &:= \max\{k\colon k\omega(n) \leq a_l\}, \quad a_{l,n} := k_{l,n}\omega(n), \\ \bar{k}_{l,n} &:= \min\{k\colon b_l \leq k\omega(n)\}, \quad b_{l,n} := \bar{k}_{l,n}\omega(n). \end{aligned}$$

Since $\omega(n) \longrightarrow 0$ we have

$$\exists n_2(\epsilon, C) > n_1(\epsilon, C)\ \forall n > n_2(\epsilon, C)\colon (a_{l,n}, b_{l,n}) \text{ disjunct for } l = 1, \ldots, u$$

and

$$\sum_{l=1}^{u} (b_{l,n} - a_{l,n}) < \epsilon \quad \text{and} \quad b_n(\bar{O}) > C \quad \text{where } \bar{O} = \bigcup_{l=0}^{u} (a_{l,n}, b_{l,n}).$$

Let $\hat{\sharp}(n)$ be the number of distinct points x_i^n in $\bar{O}$. Since in one interval $(a_{l,n}, b_{l,n})$ there are at most $\bar{k}_{l,n} - k_{l,n}$ points x_i^n, we have $\omega(n)\hat{\sharp}(n) \leq \epsilon$. But we know from Lemma 1.6 in [7] that there is a constant $\bar{c} > 0$ such that $\omega(n) > \bar{c}\beta^n$ and hence

$$\sharp(n) \leq c\beta^{-n} \quad \text{and} \quad \hat{\sharp}(n) \leq \epsilon c\beta^{-n}$$

for some constant $c > 0$. Since $b(P_n^i) = \sharp P_n^i / 2^n$ it follows from $b_n(\bar{O}) > v$ that there is a subset $\hat{\Pi}_{n,\beta}$ of $\Pi_{n,\beta}$ with $\hat{\sharp}(n)$ elements such that

$$\sum_{P \in \hat{\Pi}_{n,\beta}} \sharp P \geq C 2^n.$$

Now we estimate

$$\begin{aligned}
\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\log\beta/\log\tau} = & \sum_{P\in\hat{\Pi}_{n,\beta}} (\sharp P)^{\log\beta/\log\tau} + \sum_{P\in\Pi_{n,\beta}\setminus\hat{\Pi}_{n,\beta}} (\sharp P)^{\log\beta/\log\tau} \\
\leq & \hat{\sharp}(n)^{1-\log\beta/\log\tau}\Bigg(\sum_{P\in\hat{\Pi}_{n,\beta}} \sharp P\Bigg)^{\log\beta/\log\tau} \\
& + (\sharp(n) - \hat{\sharp}(n))^{1-\log\beta/\log\tau}\Bigg(\sum_{P\in\Pi_{n,\beta}\setminus\hat{\Pi}_{n,\beta}} \sharp P\Bigg)^{\log\beta/\log\tau} \\
\leq & (\epsilon c\beta^{-n})^{1-\log\beta/\log\tau} 2^{n\log\beta/\log\tau} + (c\beta^{-n})^{1-\log\beta/\log\tau}((1-C)2)^{n\log\beta/\log\tau} \\
= & \beta^{n(\log\beta/\log\tau - 1)} 2^{n\log\beta/\log\tau}((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau}).
\end{aligned}$$

Now choose ϵ and C such that

$$((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau}) < 1.$$

For all $n \geq n_2(\epsilon, C)$ we have

$$\begin{aligned}
& \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\log\beta/\log\tau})}{n\log\beta^{-1}} \\
< & \frac{\log(2\beta/\tau)}{\log(1/\tau)} + \frac{\log((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau})}{n\log\beta^{-1}}.
\end{aligned}$$

The last term in this sum is negative and hence our proof is complete. ∎

Proof of Theorem 2.2: From [18] we know that the box-counting dimension of $\Lambda_{\beta,\tau}$ is given by $\log(2\beta/\tau)/\log(1/\tau)$. Thus Proposition 6.1 and 6.2 immediately imply $\dim_H \Lambda_{\beta,\tau} < \dim_B \Lambda_{\beta,\tau}$ if $\beta \in (0.5, 1)$ is the reciprocal of a PV number.

This is the first statement of Theorem 2.2. Now the second statement remains to be proven. The following dimension formula for the Bernoulli measures $b^p_{\beta,\tau}$ on $\Lambda_{\beta,\tau}$ is a corollary of Theorem II of [15],

$$\dim_H b^p_{\beta,\tau} = \frac{p\log p + (1-p)\log(1-p)}{\log\tau} + \left(1 - \frac{\log\beta}{\log\tau}\right)\dim_H b^p_\beta.$$

Thus we have, by Theorem 4.1, $\dim_H b^p_{\beta,\tau} < 1$ for all $p \in (0,1)$ if $\beta \in (0.5,1)$ is the reciprocal of a PV number and τ is small enough. But, on the other hand, we have $\dim_H \Lambda_{\beta,\tau} \geq 1$ since the projection of $\Lambda_{\beta,\tau}$ on the first coordinate axis is the whole interval $[-1,1]$. This proves the second statement of our Theorem 2.2. ∎

Appendix A: Basic facts in dimension theory

In our work we denote the Hausdorff dimension of a set $Z \subseteq R^q$ by $\dim_H Z$ and the box-counting dimension (Minkovsky dimension) by $\dim_B Z$. The Hausdorff dimension of a Borel probability measure μ on $\mathbb{R}^q$ is given by

$$\dim_H \mu := \inf\{\dim_H Z | \mu(Z) = 1\}.$$

We refer to the books of Falconer [6] and Pesin [17] for the definition and the interpretation of these quantities. We summarize some basic properties of the dimensions in the following proposition.

PROPOSITION A1: *For all $Z \subseteq \mathbb{R}^q$ we have:*

(1) $\dim_H Z \leq \dim_B Z$.
(2) *If I is an interval then* $\dim_H(Z \times I) = \dim_H Z + 1$.
(3) *If f is Lipschitz then* $\dim_H f(Z) \leq \dim_H Z$.

Appendix B: Pisot–Vijayarghavan numbers

A **Pisot-Vijayarghavan number** (in brief, PV number) is by definition an algebraic integer whose algebraic conjugates all lie inside the unit circle in the complex plane. Salem [20] showed that the set of PV numbers is a closed subset of the reals and that 1 is an isolated element.

In our context we are interested in numbers $\beta \in (0.5,1)$ such that β^{-1} is a PV number. We list some examples including all reciprocals of PV numbers with minimal polynomial of degree two and three and a sequence of such numbers decreasing to 0.5.

Table 1. Reciprocals of PV numbers

x^2+x-1	$(\sqrt{5}-1)/2$
x^3+x^2+x-1	0.5436898...
x^3+x^2-1	0.754877 ...
x^3+x-1	0.6823278...
x^3-x^2+2x-1	0.5698403...
x^4-x^3-1	0.7244918...
$x^n+x^{n-1}\ldots+x-1$	$r_n \longrightarrow 0.5$

Important properties of PV numbers are that their powers are exponentially near integers (see [4]) and that the number of distinct points of the form $\sum_{k=0}^{n-1} \pm\beta^k$ is given by $\beta^{-n}+O(1)$ (see [7] and [19]). Finally, we mention that there is a book devoted to Pisot and Salem numbers [2]. Certainly the reader will find much more information about the role of these numbers in algebraic number theory and Fourier analysis in this book than we have provided here.

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